

# LEVEL 0

## O 1-10 : Tangents & Normals

### Equation of a Tangent

The equation of the tangent to the curve  $y = f(x)$  at point  $(a, f(a))$  is

$$y - f(a) = f'(a)(x - a)$$

### Equation of a Normal

The equation of the normal to the curve  $y = f(x)$  at point  $(a, f(a))$  is

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

when  $f'(a) \neq 0$ , and

$$x = a \quad \text{when } f'(a) = 0$$

Example:

$y = \frac{x}{2x+1}$ ,  $A(1, \frac{1}{3})$

Let  $f(x) = \frac{x}{2x+1}$ .

$$f'(x) = \frac{1 \cdot (2x+1) - x \cdot 2}{(2x+1)^2} = \frac{1}{(2x+1)^2}$$

$$f'(1) = \frac{1}{9}$$

Therefore, the equation of the tangent is

$$y - \frac{1}{3} = \frac{1}{9}(x - 1) \quad \therefore y = \frac{1}{9}x + \frac{2}{9}$$

The equation of the normal is  $y - \frac{1}{3} = -9(x - 1)$ .  $\therefore y = -9x + \frac{28}{3}$

When the point of tangency is not specified, define an arbitrary point on the curve and solve for it.

Given that a line passing through point  $(0, -3)$  is tangent to curve  $y = x \ln x$ , find the equation of the tangent and the coordinates of the tangent point.

Let  $f(x) = x \ln x$ .

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Let the coordinates of the tangent point be  $(a, a \ln a)$ .

The equation of the tangent is

$$y - a \ln a = (\ln a + 1)(x - a)$$

Since this line passes through  $(0, -3)$ ,

$$-3 - a \ln a = (\ln a + 1)(0 - a)$$

$$\therefore a = 3$$

Therefore, the equation of the tangent is

$$y = (\ln 3 + 1)x - 3$$

The coordinates of the tangent point are  $(3, 3 \ln 3)$ .

Sometimes, implicit differentiation is required.

$\frac{x^2}{8} + \frac{y^2}{2} = 1$ ,  $A(2, 1)$

Differentiating both sides of  $\frac{x^2}{8} + \frac{y^2}{2} = 1$  with respect to  $x$ ,

$$\frac{2x}{8} + \frac{2y}{2} \cdot y' = 0 \quad \leftarrow \frac{d}{dx} x^2 = \frac{d}{dy} y^2 \cdot \frac{dy}{dx} = 2y y'$$

Therefore, when  $y \neq 0$ ,  $y' = -\frac{x}{4y}$

Thus, the gradient of the tangent at point A is  $-\frac{2}{4 \cdot 1} = -\frac{1}{2}$ .

Therefore, the equation of the tangent is  $y - 1 = -\frac{1}{2}(x - 2)$ .

The equation of the normal is  $y - 1 = 2(x - 2)$ .

## O 11-20 : Increasing/ Decreasing Functions & Relative Extreme Values

Given that a function  $f(x)$  is continuous on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$ ,

$$f'(x) > 0 \text{ on } (a, b) \Rightarrow f(x) \text{ increases on } [a, b]$$

$$f'(x) < 0 \text{ on } (a, b) \Rightarrow f(x) \text{ decreases on } [a, b]$$

$$f'(x) = 0 \text{ on } (a, b) \Rightarrow f(x) \text{ is constant on } [a, b]$$

Note: The converses are not necessarily true.

### General tips for graphing a function

1) Determine the domain of the function.

E.g.  $y = x - 2\sqrt{x}$ , the domain is  $x \geq 0$ .

$$y = \frac{x^2}{x+1}, \text{ the domain is } x \neq -1.$$

2) Determine the asymptotes (if any).

E.g.  $y = 2x - \tan x$ , the asymptotes are  $x = \pm \frac{\pi}{2}$ .

$$y = \frac{x+1}{\sqrt{x^2+3}}, \text{ the asymptotes are } y = \pm 1.$$

Note: Graphs may cross a horizontal asymptote, but can never cross a vertical asymptote.

3) Find all stationary points (points with  $y' = 0$ ) and create the variation table.

$y = \frac{x^2}{x+1}$

The domain is  $x \neq -1$ .

$$y = x - 1 + \frac{1}{x+1}$$

$$y' = 1 - \frac{1}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

When  $y' = 0$ ,  $x = -2, 0$

Creating the variation table,

$x$	$\dots$	$-2$	$\dots$	$-1$	$\dots$	$0$	$\dots$
$y'$	$+$	$0$	$-$	$\nearrow$	$-$	$0$	$+$
$y$	$\nearrow$	$-4$	$\searrow$	$\nearrow$	$\searrow$	$0$	$\nearrow$

Therefore, the relative maximum value is  $-4$ , at  $x = -2$  and the relative minimum value is  $0$ , at  $x = 0$ .

Also,  $\lim_{x \rightarrow -1^+} y = \infty$ ,  $\lim_{x \rightarrow -1^-} y = -\infty$ ,

$$\lim_{x \rightarrow \infty} [y - (x - 1)] = 0 \text{ and } \lim_{x \rightarrow -\infty} [y - (x - 1)] = 0.$$

Thus, the asymptotes are  $x = -1$  and  $y = x - 1$ .

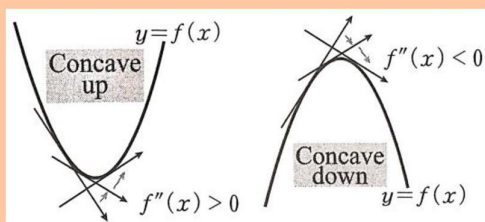
Note: If a differentiable function  $f(x)$  has a relative extremum at  $x = a$ , then  $f'(a) = 0$ . However, the converse is not true. To determine if  $f(x)$  has a relative extremum at  $x = a$ , we must look for a sign change of  $f'(x)$  as  $x$  increases through  $a$ .

### O 21-30 : Concavity of Curves

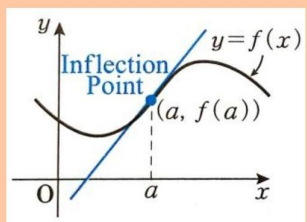
Given that a function  $f(x)$  is twice differentiable on an interval, if

$f''(x) > 0 \Rightarrow$  the curve  $y = f(x)$  is concave up.

$f''(x) < 0 \Rightarrow$  the curve  $y = f(x)$  is concave down.



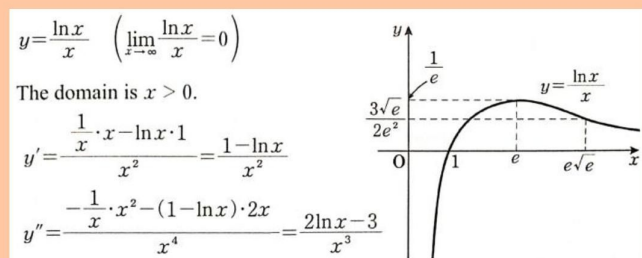
If point  $(a, f(a))$  is an inflection point of the curve  $y = f(x)$ , then  $f''(a) = 0$ . However, the converse is not true. Point  $(a, f(a))$  is an inflection point if the sign of  $f''(x)$  changes as  $x$  increases through  $a$  and  $f''(a) = 0$ .



### General Tips for graphing a function

Similar to O11-20, except that the variation table has four rows ( $x, y', y'', y$ ) instead of three rows ( $x, y', y$ ).

Example:



when  $y' = 0$ ,  $\ln x = 1$ , i.e.  $x = e$ ; and

when  $y'' = 0$ ,  $\ln x = \frac{3}{2}$ , i.e.  $x = e\sqrt{e}$

Creating the variation table,

$x$	0	...	$e$	...	$e\sqrt{e}$	...
$y'$		+	0	-	-	-
$y''$		-	-	-	0	+
$y$		$\nearrow$	$\frac{1}{e}$	$\searrow$	$\frac{3\sqrt{e}}{2e^2}$	$\searrow$

Therefore, the relative maximum value is  $\frac{1}{e}$ , at  $x = e$  and

there are no relative minimum values.

The inflection point is  $\left( e\sqrt{e}, \frac{3\sqrt{e}}{2e^2} \right)$ .  $\left[ \frac{3\sqrt{e}}{2e^2} = \frac{3}{2e\sqrt{e}} \right]$

Also, since  $\lim_{x \rightarrow 0^+} y = -\infty$  and  $\lim_{x \rightarrow \infty} y = 0$ , the asymptotes are  $x = 0$ ,  $y = 0$ .

### Second Derivative Test for Relative Extrema

Given that  $f''(x)$  is continuous on the interval including  $x = a$ , if

$\Rightarrow f'(a) = 0$  and  $f''(a) > 0$ ,  $f(a)$  represents the relative minimum value.

$\Rightarrow f'(a) = 0$  and  $f''(a) < 0$ ,  $f(a)$  represents the relative maximum value.

### O 31-40 : Maxima & Minima

To find the (global) maximum and minimum values:

1) Create the variation table for the given domain.

2) Find the relative extreme values and the values on both ends of the interval, then compare the values.

Note: When the domain is all real numbers, we need to determine the end behaviour of the graph of  $y = f(x)$ , i.e.  $\lim_{x \rightarrow \infty} y$  and  $\lim_{x \rightarrow -\infty} y$ .

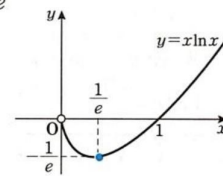
$$y = x \ln x \quad \left( \lim_{x \rightarrow 0^+} x \ln x = 0 \right)$$

The domain is  $x > 0$ .

$$y' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

When  $y' = 0$  in  $x > 0$ ,  $\ln x = -1$ , i.e.  $x = \frac{1}{e}$

$x$	0	...	$\frac{1}{e}$	...
$y'$		-	0	+
$y$		$\searrow$	$-\frac{1}{e}$	$\nearrow$



Also, since  $\lim_{x \rightarrow 0^+} y = 0$  and  $\lim_{x \rightarrow \infty} y = \infty$ , from the variation table,

there are no maximum values, and

the minimum value is  $-\frac{1}{e}$ , at  $x = \frac{1}{e}$ .

Since the domain is  $x > 0$ , determine  $\lim_{x \rightarrow 0^+} y$  and  $\lim_{x \rightarrow \infty} y$  as the values on both ends.

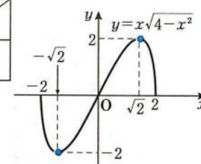
$$y = x\sqrt{4-x^2}$$

Since  $4 - x^2 \geq 0$ , the domain is  $-2 \leq x \leq 2$ .

$$y' = 1 \cdot \sqrt{4-x^2} + x \cdot \frac{-x}{\sqrt{4-x^2}} = \frac{2(x^2-2)}{\sqrt{4-x^2}}$$

When  $y' = 0$  in  $-2 < x < 2$ ,  $x = \pm\sqrt{2}$

$x$	-2	...	$-\sqrt{2}$	...	$\sqrt{2}$	...	2
$y'$		-	0	+	0	-	
$y$	0	$\searrow$	-2	$\nearrow$	2	$\searrow$	0



From the variation table,

the maximum value is 2, at  $x = \sqrt{2}$  and

the minimum value is -2, at  $x = -\sqrt{2}$ .

Refer to the worksheets for more variations.

If you have difficulty solving problems on O36-39, consult your instructor or a knowledgeable Maths tutor for additional guidance. You may also drop me a message on discord Peter Chang#4326.



## O 41-50 : Various Applications of Differentiation

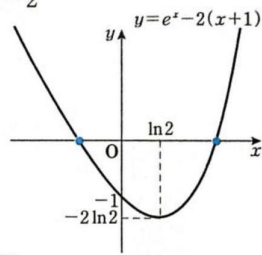
Using properties of derivatives, we can determine the number of real solutions of an equation.

$$e^x - 2(x+1) = 0 \quad \left( \lim_{x \rightarrow \infty} [e^x - 2(x+1)] = \infty \right)$$

$$\text{Let } f(x) = e^x - 2(x+1). \quad f'(x) = e^x - 2$$

When  $f'(x) = 0$ ,  $e^x = 2$ , i.e.  $x = \ln 2$

$x$	...	$\ln 2$	...
$f'(x)$	-	0	+
$f(x)$	$\searrow$	$-2\ln 2$	$\nearrow$



Also,  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $\lim_{x \rightarrow -\infty} f(x) = \infty$ .

Therefore, the equation has 2 real solutions.

### Proving important inequalities

$$e^x > 1 + x + \frac{x^2}{2} \quad (x > 0)$$

$$\text{Let } f(x) = e^x - \left(1 + x + \frac{x^2}{2}\right).$$

$$f'(x) = e^x - 1 - x$$

$$f''(x) = e^x - 1$$

Since  $f'(x) > 0$  cannot be confirmed with this expression, determine  $f''(x)$ .

When  $x > 0$ ,  $e^x > 1$ ; therefore,  $f''(x) > 0$

Thus, since  $f'(x)$  increases in  $x \geq 0$  and  $f'(0) = 0$ ,  $f'(x) > 0$

Therefore, since  $f(x)$  increases in  $x \geq 0$  and  $f(0) = 0$ ,  $f(x) > 0$

Thus, when  $x > 0$ ,  $e^x > 1 + x + \frac{x^2}{2}$

### Velocity and Acceleration

Let  $x = f(t)$  be the coordinate of point P moving on a number line at time  $t$ . The velocity  $v$  and acceleration  $a$  of point P are

$$v = \frac{dx}{dt} = f'(t) \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t)$$

### Linear Approximation I

aka 1<sup>st</sup> order Taylor Series

When the function  $f(x)$  is differentiable at  $x = a$  and the value of  $|h|$  is approaching 0, then

$$f(a+h) \approx f(a) + f'(a)h$$

### Linear Approximation II

aka 1<sup>st</sup> order Maclaurin Series

When the value of  $|x|$  is approaching 0, then

$$f(x) \approx f(0) + f'(0)x$$

## O 51-70 : Indefinite Integrals

### Indefinite Integral of $x^\alpha$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

### Properties of Indefinite Integrals

$$\int k f(x) dx = k \int f(x) dx \quad (k \text{ is a constant})$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

### Indefinite Integral of $f(ax+b)$

When  $F'(x) = f(x)$ ,  $a \neq 0$ ,

$$\int f(ax+b) dx = \frac{1}{a} F(ax+b) + C$$

Note: Sometimes, partial fraction decomposition of the integrand is necessary.

$$\begin{aligned} \int \sqrt{1-5x} dx &= \int (1-5x)^{\frac{1}{2}} dx = -\frac{2}{15} (1-5x)^{\frac{3}{2}} + C \\ &= -\frac{2}{15} (1-5x) \sqrt{1-5x} + C \end{aligned}$$

$$\begin{aligned} \int \frac{(\sqrt{x}+2)^3}{x} dx &= \int \frac{x\sqrt{x}+6x+12\sqrt{x}+8}{x} dx \\ &= \int \left( x^{\frac{1}{2}} + 6 + 12x^{-\frac{1}{2}} + \frac{8}{x} \right) dx \\ &= \frac{2}{3} x^{\frac{3}{2}} + 6x + 24x^{\frac{1}{2}} + 8\ln|x| + C \\ &= \frac{2}{3} x\sqrt{x} + 6x + 24\sqrt{x} + 8\ln x + C \end{aligned}$$

$$\begin{aligned} \int \frac{x^2+x}{x+2} dx &= \int \left( x-1 + \frac{2}{x+2} \right) dx \\ &= \frac{1}{2} x^2 - x + 2\ln|x+2| + C \\ \int \frac{3}{x^2-x-2} dx &= \int \frac{3}{(x-2)(x+1)} dx \\ &= \int \left( \frac{1}{x-2} - \frac{1}{x+1} \right) dx \\ &= \ln|x-2| - \ln|x+1| + C \\ &= \ln \left| \frac{x-2}{x+1} \right| + C \end{aligned}$$

### Indefinite Integral of Exponential Functions

$$\int e^x dx = e^x + C, \quad \int a^x dx = \frac{a^x}{\ln a} + C$$

### Indefinite Integral of Trigonometric Functions

$$\int \sin x dx = -\cos x + C, \quad \int \cos x dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} dx = \tan x + C,$$

$$\int \frac{1}{\sin^2 x} dx = -\frac{1}{\tan x} + C$$

Note: Recall all important trigonometric formulas and identities, e.g. Pythagorean Identity, Double angle identity, half angle identity etc.

$$\int \frac{e^{2x}-1}{e^x-1} dx = \int \frac{(e^x+1)(e^x-1)}{e^x-1} dx$$

$$= \int (e^x+1) dx = e^x + x + C$$

$$\int \frac{1+\cos^3 x}{\cos^2 x} dx = \int \left( \frac{1}{\cos^2 x} + \cos x \right) dx$$

$$= \tan x + \sin x + C$$

$$\int (\sin 5x + \cos 3x) dx = -\frac{1}{5} \cos 5x + \frac{1}{3} \sin 3x + C$$

$$\int (1 - \cos x)^2 dx = \int (1 - 2\cos x + \cos^2 x) dx$$

$$= \int \left[ 1 - 2\cos x + \frac{1}{2}(1 + \cos 2x) \right] dx$$

$$= \int \left( \frac{3}{2} - 2\cos x + \frac{1}{2} \cos 2x \right) dx$$

$$= \frac{3}{2}x - 2\sin x + \frac{1}{4} \sin 2x + C$$

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x (1 - \sin^2 x) \cos x dx$$

$$\text{Let } \sin x = u. \cos x dx = du$$

$$\therefore \int \sin^2 x \cos^3 x dx = \int u^2 (1 - u^2) du$$

$$= \int (u^2 - u^4) du$$

$$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

Useful Formula:

$$\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$$

$$\int \frac{4x^3}{x^4+1} dx = \int \frac{(x^4+1)'}{x^4+1} dx = \ln|x^4+1| + C = \ln(x^4+1) + C$$

$$\int \frac{x-1}{x^2-2x-3} dx = \int \frac{(x^2-2x-3)'}{x^2-2x-3} \cdot \frac{1}{2} dx = \frac{1}{2} \ln|x^2-2x-3| + C$$

### O 71-80 : Integration by Substitution

$$\int f(x) dx = \int f(g(t))g'(t) dt \quad (x = g(t))$$

$$\int (x-2)\sqrt{3-x} dx$$

$$\text{Let } \sqrt{3-x} = t. \text{ Since } 3-x = t^2, x = 3-t^2 \therefore dx = -2t dt$$

$$\therefore \int (x-2)\sqrt{3-x} dx = \int (1-t^2)t \cdot (-2t) dt$$

$$= 2 \int (t^4 - t^2) dt$$

$$= 2 \left( \frac{1}{5} t^5 - \frac{1}{3} t^3 \right) + C$$

$$= \frac{2}{15} t^3 (3t^2 - 5) + C$$

$$= \frac{2}{15} (4-3x)(3-x)\sqrt{3-x} + C$$

$$\int f(g(x))g'(x) dx = \int f(u) du \quad (g(x) = u)$$

$$\int \frac{x^3}{\sqrt{x^2+1}} dx$$

$$\text{Let } \sqrt{x^2+1} = u. \text{ Since } x^2+1 = u^2, 2x dx = 2u du$$

$$\therefore x dx = u du$$

$$\therefore \int \frac{x^3}{\sqrt{x^2+1}} dx = \int \frac{u^2-1}{u} \cdot u du$$

$$= \int (u^2 - 1) du$$

$$= \frac{1}{3} u^3 - u + C$$

$$= \frac{1}{3} (x^2+1)^{3/2} - \sqrt{x^2+1} + C$$

$$\int \frac{(\ln x)^2}{x} dx$$

$$\text{Let } \ln x = u. \frac{1}{x} dx = du$$

$$\therefore \int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} (\ln x)^3 + C$$

### O 81-90 : Integration by Parts

#### Integration by Parts

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

How to choose  $f(x)$  in integration by parts

We choose  $f(x)$  based on the LATE rule (1 being the first choice and 4 being the last choice)

- 1) **L** – Logarithmic function, e.g.  $\ln x$
- 2) **A** – Algebraic expression, e.g.  $x^2, \sqrt{x}$
- 3) **T** – Trigonometric function, e.g.  $\sin x$
- 4) **E** – Exponential function, e.g.  $e^x$

$$\int x^4 \ln x dx = \int \left( \frac{1}{5} x^5 \right)' \ln x dx \quad \text{Using the LATE rule,}$$

$$\text{let } f(x) = \ln x, g'(x) = x^4$$

$$= \frac{1}{5} x^5 \ln x - \int \frac{1}{5} x^5 (\ln x)' dx$$

$$= \frac{1}{5} x^5 \ln x - \frac{1}{5} \int x^4 dx = \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + C$$

$$\int x \sin x dx = \int x (-\cos x)' dx \quad \text{Using the LATE rule,}$$

$$\text{let } f(x) = x, g'(x) = \sin x$$

$$= -x \cos x - \int (x)' (-\cos x) dx$$

$$= -x \cos x + \int \cos x dx = -x \cos x + \sin x + C$$

$$\int x e^{2x} dx = \int x \left( \frac{1}{2} e^{2x} \right)' dx \quad \text{Using the LATE rule,}$$

$$\text{let } f(x) = x, g'(x) = e^{2x}$$

$$= \frac{1}{2} x e^{2x} - \int (x)' \cdot \frac{1}{2} e^{2x} dx$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

Note: In some cases, one might need to perform integration by parts twice (or multiple times).

$$\int (\ln x)^2 dx = \int (x')' (\ln x)^2 dx$$

$$= x(\ln x)^2 - \int x[(\ln x)^2]' dx$$

$$= x(\ln x)^2 - 2 \int \ln x dx$$

$$= x(\ln x)^2 - 2 \int (x')' \ln x dx$$

$$= x(\ln x)^2 - 2 \left[ x \ln x - \int x(\ln x)' dx \right]$$

$$= x(\ln x)^2 - 2x \ln x + 2 \int dx$$

$$= x(\ln x)^2 - 2x \ln x + 2x + C$$

$$\begin{aligned} \int x[(\ln x)^2]' dx \\ = \int x \cdot 2 \ln x \cdot \frac{1}{x} dx \\ = 2 \int \ln x dx \end{aligned}$$

$$\int e^x \cos x dx = \int (e^x)' \cos x dx$$

Using the LATE rule  
 $f(x) = \cos x$ ,  $g'(x) = e^x$

$$= e^x \cos x - \int e^x (\cos x)' dx$$

$$= e^x \cos x + \int e^x \sin x dx$$

$$= e^x \cos x + \int (e^x)' \sin x dx$$

$$= e^x \cos x + \left[ e^x \sin x - \int e^x (\sin x)' dx \right]$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x dx$$

$$\therefore 2 \int e^x \cos x dx = e^x (\cos x + \sin x) + C_1$$

Considering the constant of integration,

$$\int e^x \cos x dx = \frac{1}{2} e^x (\cos x + \sin x) + C$$

## O 91-100 : Definite Integrals

If  $F(x)$  is an indefinite integral of  $f(x)$ , then

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x} + \sqrt{x+1}} &= \int_0^1 \frac{\sqrt{x} - \sqrt{x+1}}{(\sqrt{x} + \sqrt{x+1})(\sqrt{x} - \sqrt{x+1})} dx \\ &= \int_0^1 (\sqrt{x+1} - \sqrt{x}) dx \\ &= \left[ \frac{2}{3} (x+1)^{\frac{3}{2}} - \frac{2}{3} x^{\frac{3}{2}} \right]_0^1 \\ &= \left( \frac{2}{3} \cdot 2^{\frac{3}{2}} - \frac{2}{3} \cdot 1^{\frac{3}{2}} \right) - \frac{2}{3} \cdot 1^{\frac{3}{2}} = \frac{4}{3} (\sqrt{2} - 1) \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{dx}{(x+1)(x+2)} &= \int_0^1 \left( \frac{1}{x+1} - \frac{1}{x+2} \right) dx \\ &= [\ln|x+1| - \ln|x+2|]_0^1 \\ &= \left[ \ln \left| \frac{x+1}{x+2} \right| \right]_0^1 = \ln \frac{2}{3} - \ln \frac{1}{2} = \ln \frac{4}{3} \end{aligned}$$

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x dx &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \cos 2x) dx \\ &= \frac{1}{2} \left[ x - \frac{1}{2} \sin 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \frac{1}{2} \left[ \frac{\pi}{2} - \left( \frac{\pi}{4} - \frac{1}{2} \right) \right] = \frac{\pi}{8} + \frac{1}{4} \end{aligned}$$

## Property of Definite Integrals I

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\begin{aligned} \int_0^{2\pi} |\sin x| dx &= \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} (-\sin x) dx \\ &= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} \\ &= [1 - (-1)] + [1 - (-1)] = 4 \end{aligned}$$

## O 101-110 : Integration by Substitution

When  $x = g(t)$ , if  $a = g(\alpha)$  and  $b = g(\beta)$ , then

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(g(t)) g'(t) dt$$

$$\int_0^1 \frac{x^3}{1+x^2} dx \quad \begin{array}{c|c} x & 0 \longrightarrow 1 \\ t & 1 \longrightarrow 2 \end{array}$$

Let  $1+x^2=t$ . Since  $x^2=t-1$ ,  $2x dx=dt$

$$\begin{aligned} \therefore \int_0^1 \frac{x^3}{1+x^2} dx &= \int_1^2 \frac{t-1}{t} \cdot \frac{1}{2} dt \\ &= \frac{1}{2} \int_1^2 \left( 1 - \frac{1}{t} \right) dt \\ &= \frac{1}{2} [t - \ln|t|]_1^2 = \frac{1}{2} (1 - \ln 2) \end{aligned}$$

## Property of Definite Integrals II

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$

$$\begin{aligned} \int_0^2 x \sqrt{4-x^2} dx &= \int_0^2 x \sqrt{4-x^2} dx \\ \text{Let } \sqrt{4-x^2}=t. \text{ Since } 4-x^2=t^2, -2x dx=2t dt \therefore x dx=-t dt \\ \therefore \int_0^2 x \sqrt{4-x^2} dx &= \int_2^0 t \cdot (-t) dt \\ &= \int_0^2 t^2 dt = \left[ \frac{1}{3} t^3 \right]_0^2 = \frac{8}{3} \end{aligned}$$

## Trigonometric Substitution:

Let  $x = a \sin \theta$  for the definite integral of  $\sqrt{a^2 - x^2}$

Let  $x = a \tan \theta$  for the definite integral of  $\frac{1}{x^2 + a^2}$

$$\begin{aligned} \int_0^2 \frac{dx}{\sqrt{16-x^2}} &= \int_0^2 \frac{dx}{\sqrt{16-x^2}} \\ \text{Let } x=4 \sin \theta. dx=4 \cos \theta d\theta & \quad \begin{array}{c|c} x & 0 \longrightarrow 2 \\ \theta & 0 \longrightarrow \frac{\pi}{6} \end{array} \\ \text{When } 0 \leq \theta \leq \frac{\pi}{6}, \cos \theta > 0; \text{ therefore,} & \\ \sqrt{16-x^2} &= \sqrt{16(1-\sin^2 \theta)} = 4 \cos \theta \\ \therefore \int_0^2 \frac{dx}{\sqrt{16-x^2}} &= \int_0^{\frac{\pi}{6}} \frac{1}{4 \cos \theta} \cdot 4 \cos \theta d\theta = \int_0^{\frac{\pi}{6}} d\theta = [\theta]_0^{\frac{\pi}{6}} = \frac{\pi}{6} \end{aligned}$$



### Integration of Even/Odd Functions

$f(x)$  is even function,  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

$f(x)$  is odd function,  $\int_{-a}^a f(x) dx = 0$

$$\int_{-e}^e x e^{x^2} dx$$

Let  $f(x) = x e^{x^2}$ .  $f(-x) = -x e^{(-x)^2} = -x e^{x^2} = -f(x)$   
 $f(x)$  is an odd function.

$$\therefore \int_{-e}^e x e^{x^2} dx = 0$$

### O 111-120 : Integration by Parts & Functions Represented by Definite Integrals

We use the LATE rule in integration by parts

$$\begin{aligned} \int_e^{2e} \ln x dx &= \int_e^{2e} (x)' \ln x dx \quad \leftarrow \int \ln x dx = \int 1 \cdot \ln x dx \\ &= [x \ln x]_e^{2e} - \int_e^{2e} x (\ln x)' dx \\ &= 2e \ln 2 + e - \int_e^{2e} dx \quad \leftarrow \begin{matrix} \ln 2e \\ = \ln 2 + \ln e \\ = \ln 2 + 1 \end{matrix} \\ &= 2e \ln 2 + e - [x]_e^{2e} = 2e \ln 2 \end{aligned}$$

$$\begin{aligned} \int_{-1}^1 x^2 e^{1-x} dx &= \int_{-1}^1 x^2 (-e^{1-x})' dx \\ &= [-x^2 e^{1-x}]_{-1}^1 - \int_{-1}^1 (x^2)' (-e^{1-x}) dx \\ &= -1 + e^2 + 2 \int_{-1}^1 x e^{1-x} dx \\ &= -1 + e^2 + 2 \int_{-1}^1 x (-e^{1-x})' dx \\ &= -1 + e^2 + 2 \left\{ [-x e^{1-x}]_{-1}^1 - \int_{-1}^1 (x)' (-e^{1-x}) dx \right\} \\ &= -e^2 - 3 + 2 \int_{-1}^1 e^{1-x} dx \\ &= -e^2 - 3 + 2 [-e^{1-x}]_{-1}^1 = e^2 - 5 \end{aligned}$$

### Fundamental Theorem of Calculus

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

$$\begin{aligned} F(x) &= \int_0^x (x-t) \cos t dt \quad \left\{ \begin{matrix} x \text{ is not affected by the variable of integration } t \text{ and} \\ \text{can therefore be taken to the front of } \int. \end{matrix} \right. \\ F(x) &= x \int_0^x \cos t dt - \int_0^x t \cos t dt \quad \left\{ \begin{matrix} f(x)=x, g(x)=\int_0^x \cos t dt \\ [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \end{matrix} \right. \\ F'(x) &= 1 \cdot \int_0^x \cos t dt + x \left( \frac{d}{dx} \int_0^x \cos t dt \right) - x \cos x \quad \leftarrow \\ &= \int_0^x \cos t dt + x \cos x - x \cos x \quad \left\{ \frac{d}{dx} \int_a^x f(t) dt = f(x) \right. \\ &= [\sin t]_0^x \\ &= \sin x \end{aligned}$$

### Property of Definite Integrals III

$$\int_a^a f(x) dx = 0$$

### O 121-130 : Integration by Quadrature & Proof of Inequalities

If  $f(x)$  is continuous on the interval  $[a, b]$ ,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} f(x_k) \Delta x = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x = \int_a^b f(x) dx$$

where  $\Delta x = \frac{b-a}{n}$ ,  $x_k = a + k \Delta x$

#### Important result:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$$

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \dots + \frac{1}{1+\frac{n}{n}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1+\frac{k}{n}} \\ &= \int_0^1 \frac{dx}{1+x} = [\ln |1+x|]_0^1 = \ln 2 \end{aligned}$$

When proving inequalities, we need to use the following two results:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n-1}$$

Let  $k$  be a natural number. When  $k \leq x \leq k+1$ ,  $\frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{x}}$

Also, the equality sign does not hold for all values of  $x$ .

$$\begin{aligned} \therefore \int_k^{k+1} \frac{dx}{\sqrt{k+1}} &< \int_k^{k+1} \frac{dx}{\sqrt{x}} \\ \therefore \frac{1}{\sqrt{k+1}} &< \int_k^{k+1} \frac{dx}{\sqrt{x}} \quad \dots \textcircled{1} \end{aligned}$$

Substituting  $k=1, 2, 3, \dots, n-1$  into  $\textcircled{1}$  and adding up the terms on each side, when  $n \geq 2$ ,

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt{k+1}} < \sum_{k=1}^{n-1} \int_k^{k+1} \frac{dx}{\sqrt{x}}$$

$$\text{LHS} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{n}}$$

$$\begin{aligned} \text{RHS} &= \int_1^2 \frac{dx}{\sqrt{x}} + \int_2^3 \frac{dx}{\sqrt{x}} + \int_3^4 \frac{dx}{\sqrt{x}} + \dots + \int_{n-1}^n \frac{dx}{\sqrt{x}} \\ &= \int_1^n \frac{dx}{\sqrt{x}} \\ &= [2x^{\frac{1}{2}}]_1^n = 2\sqrt{n} - 2 \end{aligned}$$

$$\therefore \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2$$

Adding 1 to both sides,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1$$

$\textcircled{1}$  On the interval  $[a, b]$ , if  $f(x) \geq g(x)$ , then

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx$$

Equality only when  $f(x) = g(x)$  for all values of  $x$

② If  $a_n \geq b_n$  for  $n = 1, 2, \dots, N$ , then

$$\sum_{n=1}^N a_n \geq \sum_{n=1}^N b_n$$

Equality only when  $a_n = b_n$  for  $n \geq 1$

### Cauchy-Schwarz Inequality

$$\left[ \int_a^b f(x)g(x) dx \right]^2 \leq \left( \int_a^b [f(x)]^2 dx \right) \left( \int_a^b [g(x)]^2 dx \right)$$

### O 131-140 : Areas

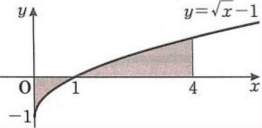
Given area  $S$  enclosed by the curve  $y = f(x)$ , the  $x$ -axis and two lines  $x = a$  and  $x = b$  ( $a < b$ ),

- When  $f(x) \geq 0$  on  $[a, b]$ ,  $S = \int_a^b f(x) dx$

- When  $f(x) \leq 0$  on  $[a, b]$ ,  $S = -\int_a^b f(x) dx$

$$y = \sqrt{x} - 1, \quad x = 0, \quad x = 4$$

Since  $y \leq 0$  in  $0 \leq x \leq 1$  and  $y \geq 0$  in  $1 \leq x \leq 4$ ,

$$\begin{aligned} S &= -\int_0^1 (\sqrt{x} - 1) dx + \int_1^4 (\sqrt{x} - 1) dx \\ &= -\left[ \frac{2}{3} x^{\frac{3}{2}} - x \right]_0^1 + \left[ \frac{2}{3} x^{\frac{3}{2}} - x \right]_1^4 = 2 \end{aligned}$$


On the interval  $[a, b]$ , when  $f(x) \geq g(x)$ , the area between the curves  $y = f(x)$  and  $y = g(x)$

$$S = \int_a^b [f(x) - g(x)] dx$$

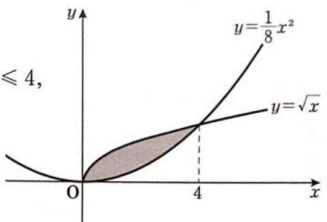
Find the area  $S$  enclosed by the following two curves.

$$f(x) = \frac{1}{8}x^2, \quad g(x) = \sqrt{x}$$

Since  $\frac{1}{8}x^2 = \sqrt{x}$ , the  $x$ -coordinates of the points of intersection are

$$\begin{aligned} x^4 - 64x &= 0 \\ \therefore x &= 0, 4 \end{aligned}$$

Since  $f(x) \leq g(x)$  in  $0 \leq x \leq 4$ ,

$$\begin{aligned} S &= \int_0^4 \left( \sqrt{x} - \frac{1}{8}x^2 \right) dx \\ &= \left[ \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{24} x^3 \right]_0^4 = \frac{8}{3} \end{aligned}$$


Given area  $S$  enclosed by the curve  $x = g(y)$ , the  $y$ -axis and two lines  $y = c$  and  $y = d$  ( $c < d$ ),

- When  $g(y) \geq 0$  on  $[c, d]$ ,  $S = \int_c^d g(y) dy$

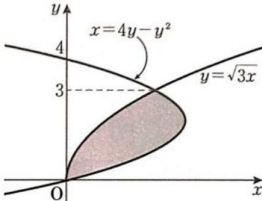
Find the area  $S$  enclosed by two curves  $x = 4y - y^2$  and  $y = \sqrt{3x}$ .

$$\text{Since } y = \sqrt{3x}, \quad x = \frac{1}{3}y^2$$

$$\text{When } 4y - y^2 = \frac{1}{3}y^2,$$

$$\therefore y = 0, 3$$

Since  $4y - y^2 \geq \frac{1}{3}y^2$  in  $0 \leq y \leq 3$ ,

$$\begin{aligned} S &= \int_0^3 \left( 4y - y^2 - \frac{1}{3}y^2 \right) dy \\ &= \int_0^3 \left( 4y - \frac{4}{3}y^2 \right) dy = \left[ 2y^2 - \frac{4}{9}y^3 \right]_0^3 = 6 \end{aligned}$$


Sometimes, curves are expressed in parametric form

Find the area enclosed by the  $x$ -axis and the curve represented by  $x = t - 1$ ,  $y = 2t - t^2$  in  $0 \leq t \leq 2$ .

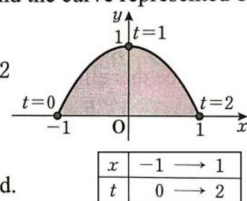
When  $y = 0$ , since  $t(2 - t) = 0$ ,  $t = 0, 2$

Also,  $y \geq 0$  in  $0 \leq t \leq 2$ .

Then, since  $x = t - 1$ ,  $dx = dt$

Therefore, let  $S$  be the area to be found.

$$S = \int_{-1}^1 y dx = \int_0^2 (2t - t^2) dt = \left[ t^2 - \frac{1}{3}t^3 \right]_0^2 = \frac{4}{3}$$



### O 141-150 : Volumes

Let  $S(x)$  be the cross-sectional area of a solid cut by the plane perpendicular to the  $x$ -axis and intersecting the  $x$ -axis at  $x$ , then the volume  $V$  in  $a \leq x \leq b$  is

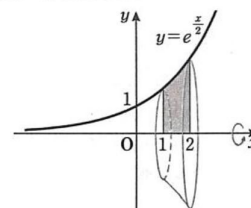
$$V = \int_a^b S(x) dx$$

### Volume of Revolution about the $x$ -axis

$$V = \pi \int_a^b [f(x)]^2 dx = \pi \int_a^b y^2 dx$$

Given a solid formed by rotating the area enclosed by curve  $y = e^{\frac{x}{2}}$ , the  $x$ -axis and two lines  $x = 1$  and  $x = 2$  once about the  $x$ -axis, find the volume  $V$  of this solid.

$$\begin{aligned} V &= \pi \int_1^2 e^x dx \\ &= \pi [e^x]_1^2 \\ &= \pi e(e - 1) \end{aligned}$$

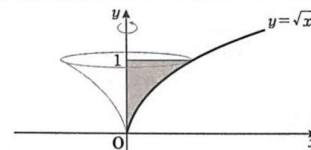


### Volume of Revolution about the $y$ -axis

$$V = \pi \int_c^d [g(y)]^2 dy = \pi \int_c^d x^2 dy$$

Given a solid formed by rotating the area enclosed by curve  $y = \sqrt{x}$ , the  $y$ -axis and line  $y = 1$  once about the  $y$ -axis, find the volume  $V$  of this solid.

$$\begin{aligned} \text{Since } x &= y^2, \\ V &= \pi \int_0^1 y^4 dy \\ &= \pi \left[ \frac{1}{5} y^5 \right]_0^1 = \frac{\pi}{5} \end{aligned}$$



Given a solid formed by rotating the area enclosed by curve  $y = \sqrt[3]{x}$  and line  $y = x$  once about the  $x$ -axis, find the volume  $V$  of this solid. ( $x \geq 0$ )

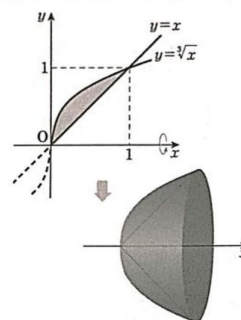
$$\text{When } \sqrt[3]{x} = x, \quad x^3 - x = 0$$

$$x(x+1)(x-1) = 0$$

Since  $x \geq 0$ ,  $x = 0, 1$

Since  $\sqrt[3]{x} \geq x$  in  $0 \leq x \leq 1$ ,

$$\begin{aligned} V &= \pi \int_0^1 (\sqrt[3]{x})^2 dx - \pi \int_0^1 x^2 dx \\ &= \pi \int_0^1 \left( x^{\frac{2}{3}} - x^2 \right) dx \\ &= \pi \left[ \frac{3}{5} x^{\frac{5}{3}} - \frac{1}{3} x^3 \right]_0^1 = \frac{4}{15} \pi \end{aligned}$$



Refer to the worksheets for more variations.



## O 151-160 : Length of a Curve, Velocity & Distance

### Length of a Curve

The length  $L$  of the curve  $x = f(t)$ ,  $y = g(t)$  ( $a \leq t \leq b$ ) is

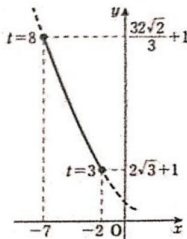
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

If instead we have  $y = f(x)$  ( $a \leq x \leq b$ ), then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Given that the coordinates of point  $P(x, y)$  moving on the plane at time  $t$  are  $x = -t + 1$  and  $y = \frac{2}{3}t\sqrt{t} + 1$ , find the distance  $l$  travelled by point  $P$  from  $t = 3$  to  $t = 8$ .

$$\begin{aligned}\frac{dx}{dt} &= -1, \quad \frac{dy}{dt} = \sqrt{t} \\ \therefore l &= \int_3^8 \sqrt{(-1)^2 + (\sqrt{t})^2} dt \\ &= \int_3^8 \sqrt{1+t} dt \\ &= \left[ \frac{2}{3}(1+t)^{\frac{3}{2}} \right]_3^8 = \frac{38}{3}\end{aligned}$$



Let  $x = f(t)$  be the coordinate of point  $P$  moving on a number line at time  $t$ , and let  $v$  be its velocity. Therefore, the displacement  $s$  and the distance  $l$  of point  $P$  from  $t = a$  to  $t = b$  are given as

$$s = \int_a^b v dt, \quad l = \int_a^b |v| dt$$

Given that the velocity  $v$  of point  $P$  moving on a number line at time  $t$  is  $1 - \sqrt{t}$ , find the displacement  $s$  of point  $P$  and the distance  $l$  travelled by point  $P$  from  $t = 0$  to  $t = 4$ .

$$\begin{aligned}s &= \int_0^4 (1 - \sqrt{t}) dt = \left[ t - \frac{2}{3}t^{\frac{3}{2}} \right]_0^4 = -\frac{4}{3} \\ \text{Also, } 1 - \sqrt{t} &\geq 0 \text{ when } 0 \leq t \leq 1 \\ 1 - \sqrt{t} &\leq 0 \text{ when } 1 \leq t \leq 4 \\ \therefore l &= \int_0^1 (1 - \sqrt{t}) dt + \int_1^4 [-(1 - \sqrt{t})] dt \\ &= \left[ t - \frac{2}{3}t^{\frac{3}{2}} \right]_0^1 + \left[ -t + \frac{2}{3}t^{\frac{3}{2}} \right]_1^4 = 2\end{aligned}$$

Refer to the worksheets for more applications.

## O 161-170 : Differential Equations

We use method of separating variables if the differential equation has form  $f(y) \frac{dy}{dx} = g(x)$ .

$$\frac{dy}{dx} = y \cos x \quad \dots \textcircled{1}$$

(i) It is obvious that constant function  $y = 0$  is a solution.

(ii) When  $y \neq 0$ , rearranging  $\textcircled{1}$ ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \quad \int \frac{dy}{y} = \int \cos x dx$$

$$\therefore \ln|y| = \sin x + C_1$$

$$\therefore y = \pm e^{\sin x + C_1} = \pm e^{C_1} \cdot e^{\sin x}$$

Let  $\pm e^{C_1} = C$ .  $y = Ce^{\sin x}$  ( $C$  is an arbitrary constant,  $C \neq 0$ )

When  $C = 0$  in (ii),  $y = 0$ , which is the same as (i).

$$\therefore y = Ce^{\sin x} \quad (C \text{ is an arbitrary constant})$$

If an initial condition is given, solve for the arbitrary constant in the general solution

$$\sqrt{1+x} \frac{dy}{dx} = \sqrt{1+y} \quad \dots \textcircled{1} \quad [\text{When } x=3, y=8]$$

Constant functions  $x = -1$  and  $y = -1$  do not satisfy the given condition.

When  $x \neq -1$  and  $y \neq -1$ , rearranging  $\textcircled{1}$ ,

$$\begin{aligned}\frac{1}{\sqrt{1+y}} \cdot \frac{dy}{dx} &= \frac{1}{\sqrt{1+x}} & \int \frac{dy}{\sqrt{1+y}} &= \int \frac{dx}{\sqrt{1+x}} \\ \therefore 2\sqrt{1+y} &= 2\sqrt{1+x} + C\end{aligned}$$

When  $x = 3$ ,  $y = 8$ ; therefore,  $C = 2$

$$\therefore \sqrt{1+y} = \sqrt{1+x} + 1$$

Solving using substitution is a clever trick

$$\frac{dy}{dx} = \frac{1-x-y}{x+y} \quad \dots \textcircled{1}$$

Differentiating both sides of  $x + y = u$  with respect to  $x$ ,

$$1 + \frac{dy}{dx} = \frac{du}{dx} \quad \dots \textcircled{2}$$

From  $\textcircled{1}$  and  $\textcircled{2}$ ,  $\frac{du}{dx} = 1 + \frac{1-u}{u} = \frac{1}{u}$ , i.e.  $u \frac{du}{dx} = 1$

$$\int u du = \int dx \quad \therefore \frac{1}{2}u^2 = x + C_1$$

$$\therefore (x+y)^2 = 2x + 2C_1$$

Let  $2C_1 = C$ .

$$(x+y)^2 = 2x + C \quad (C \text{ is an arbitrary constant})$$

## O 171-180 : Applications of Differential Equations

This particular set includes applications of differential equations in Natural/ Social Science. The following provides explanation on a few differential equations from O176, 177

**O176a:**  $L \frac{dx}{dt}$  is the voltage drop due to an inductor,  $Rx$  is the voltage drop due to a resistor (from Ohm's Law). The differential equation  $L \frac{dx}{dt} + Rx = V$  is derived from one of the Kirchhoff's Laws.

**O176b:**  $mg - kv$  is the resultant force acting on the raindrop (in the direction towards the ground). The differential equation  $m \frac{dv}{dt} = mg - kv$  is a result of Newton's second law of motion.

**O177a:**  $A \frac{dx}{dt}$  is the instantaneous rate of change in water volume, and that is equal to the area of the exit hole  $a$  multiplied by the rate of water outflow  $\sqrt{2gx}$  (from Torricelli's Law).

## O 181-200 : Applications of Calculus

This particular set includes **challenging problems** that require higher order thinking skills. Make sure to go through the problems (and solutions) *carefully* with your instructor.

If you have difficulty solving them, you may drop me a message on discord Peter Chang#4326.